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# A fractional optimal control problem for maximizing advertising efficiency

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# A fractional optimal control problem for maximizing advertising efficiency *

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**Abstract.** We propose an optimal control problem to model the dynamics of the communication activity of a firm with the aim of maximizing its efficiency. We assume that the advertising effort undertaken by the firm contributes to increase the firm's goodwill and that the goodwill affects the firm's sales. The aim is to find the advertising policies in order to maximize the firm's efficiency index which is computed as the ratio between "outputs" and "inputs" properly weighted; the outputs are represented by the final level of goodwill and by the sales achieved by the firm during the period considered, whereas the inputs are represented by the costs undertaken by the firm, fixed costs and advertising costs. The problem considered is formulated as a fractional optimal control problem. In order to find the optimal advertising policies we use the Dinkelbach's algorithm, for fractional programming.

**Keywords:** optimal control, advertising, efficiency, fractional programming.

**JEL Classification Numbers:** C61, M37.

**MathSci Classification Numbers:** 90B60, 49N05.

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## 1 Introduction

In this paper we present a fractional optimal control problem which models the dynamics of communication activity of a firm with the aim of maximizing advertising efficiency.

The problem of determining the optimal communication policy undertaken by a firm has been largely analyzed in marketing literature by means of dynamic optimal control models (see [7] and [5], for a review). The various advertising models essentially differ from each other in the dynamics which connects advertising to sales and in the objectives pursued by the firm.

We assume here that the dynamics is the same as the one in the well known Nerlove-Arrow model, in which advertising is considered as an investment and the advertising capital (the concept of goodwill) takes into account of the long term effects of advertising on consumers' demand (see [6]). Moreover, as in the classical capital advertising models, we assume that the rate of sales depends on the stock of goodwill.

Nevertheless, unlike the Nerlove-Arrow model and unlike other advertising models, we consider a special objective functional that represents the efficiency of the firm.

More precisely, we assume that the firm aims at reaching simultaneously the following objectives, in a given time period:

i) maximization of total sales,

ii) maximization of final level of goodwill,

iii) minimization of total costs.

In many advertising models the objectives i) and iii) are simultaneously taken into account in building the firm's profit function and the optimal control problem consists in maximizing either the net profit over a finite horizon, or the present value of the net profit in case of infinite time horizon. Examples of such functionals can be found in [6], [7]. On the other hand, some other models consider as the unique goal the maximization of sales, or the minimization of the total expenditure in communication, this occurs for instance in [2].

Differently, we consider a special efficiency index to be maximized. The concept of technical efficiency can be seen as a ratio between the output produced by the firm and the input used in the production process (a sort of productivity ratio). Since in our paper we focus on the communication process, we deliberately do not consider some aspects associated to the production process, such as the variable production costs. We only concentrate on the relations connecting the advertising expenditure rate to the advertising capital and the impact of goodwill on sales. In this context we can see the total sales obtained by the firm during the time interval considered as a relevant output. Moreover, since it appears better to achieve an high level of goodwill at the end of the selling period, indicating possible larger sales for the future, we include among the outputs also the final level of goodwill. The inputs, which should be minimized, consist in the costs undertaken by the firm, fixed costs and total advertising costs. This way the problem of reaching the maximum efficiency allows to take into account the three above mentioned objectives at the same time.

Given the fractional nature of the efficiency index, the problem considered is formulated as a fractional optimal control problem, for the resolution of which we cannot directly use the standard optimal control theory. We propose to resort to the algorithm by Dinkelbach for fractional programming, which allows to obtain a solution to the original fractional problem by studying associated linear control problems.

The paper is organized as follows. In Section 2 we formulate the efficiency maximization problem which drives to a fractional optimal control problem. In Section 3 we present the Dinkelbach's approach for fractional programming problems and discuss the optimal advertising policies, whereas in Section 4 we present the algorithm, some sensitivity analysis results and a numerical example. Some conclusive remarks are given in Section 5 while the proofs of the propositions are reported in the Appendix.

# 2 The efficiency maximization problem

We consider the communication activity of a firm in a limited selling period $[0, T]$ and assume that communication is performed only by means of advertising. Let us denote by

$a(t)$ = the advertising expenditure rate at time $t$,

$A(t)$ = the goodwill level at time $t$,

$S(t)$ = the rate of sales at time $t$.

We consider the following differential equation for the goodwill dynamics

$$\dot{A}(t) = -\delta A(t) + \epsilon a(t) \tag{1}$$

with the initial condition

$$A(0) = A_0 \tag{2}$$

We note that equation (1) is the same as in the Nerlove-Arrow model, apart from the parameter $\epsilon > 0$ that represents the advertising productivity in terms of goodwill.
The efficiency index is build as the ratio between outputs and inputs properly weighted. The outputs are represented by the final level of goodwill $A(T)$ and by the sales achieved by the firm during the period considered $\int_0^T S(t)dt$, whereas the inputs are represented by the fixed costs $C_0$ and by the total advertising costs $\int_0^T a(t)dt$.
The efficiency index thus, is computed as follows:

$$EI = \frac{\alpha A(T) + (1-\alpha)\int_0^T S(t)dt}{C_0 + \int_0^T a(t)dt} \tag{3}$$

where $\alpha \in (0, 1)$ represents the weight of total goodwill.
If we put $k = (1 - \alpha)/\alpha$ we can rewrite the efficiency index as

$$EI = \alpha \frac{A(T) + k\int_0^T S(t)dt}{C_0 + \int_0^T a(t)dt} \tag{4}$$

This way the parameter $k > 0$ represents the weight assigned to the total sales. In agreement with Favaretto and Viscolani [4], we assume that the sales rate S(t) is an affine transformation of the goodwill level, as follows

$$S(t) = A(t) + b\ , \quad b \geq 0\ . \tag{5}$$

It is not restrictive to assume the constancy of product sell price since the selling period considered is short enough so that a linear model can be seen a sufficiently good approximation of reality.

The efficiency maximization problem consists in maximizing the efficiency index $EI$ under the constraints (1), (2) and assuming a selling function (5):

$$\begin{array}{ll} FP\ :\ \max & \dfrac{A(T) + k\int_0^T (A(t)+b)\ dt}{C_0 + \int_0^T a(t)dt} \\ \text{subject to} & \dot{A}(t) = -\delta A(t) + \epsilon a(t) \\ & A(0) = A_0 \\ & a \in [0,\, \overline{a}], \end{array}$$

where $\overline{a} > 0$, is the maximum advertising expenditure rate.

Problem $FP$ has one state variable $A(t)$, continuous and piecewise continuously differentiable, and one control variable $a(t)$, piecewise continuous. The maximum efficiency problem is a linear fractional optimal control problem with a finite horizon, for which resolution we cannot directly use the standard optimal control theory.

We propose to resort to the algorithm by Dinkelbach [3] for fractional programming, which will be presented in the next section.

## 3 Dinkelbach's approach and optimal advertising policies

A possible way to solve problem $FP$ is to use Dinkelbach's algorithm as modified by Bhatt [1] and Stancu-Minasian [9] for fractional optimal control problems.

The approach consists in a sort of linearization of the objective functional. More precisely let us define for each $q \in \mathbb{R}$ the auxiliary function $F(q)$ whose value is the maximum value of the optimal control problem:

$$P_q : \max_{(A,a)\in\Omega} \left[ \left( A(T) + k\int_0^T (A(t)+b)dt \right) - q\left( C_0 + \int_0^T a(t)dt \right) \right] \tag{6}$$

$$\tag{7}$$

where $\Omega$ is defined by the dynamic system

$$\dot{A}(t) = -\delta A(t) + \epsilon a(t), \tag{8}$$

$$A(0) = A_0, \tag{9}$$

$$a \in [0,\, \overline{a}] \tag{10}$$

Remark that, for each fixed $q$, $P_q$ can be solved by classical linear optimal control techniques.

It is possible to prove that function $F$ is strictly decreasing and convex and has a (unique) zero $q^*$ (see [9]). Moreover, a nice property relates the original fractional optimal control problem $FP$ with the set of auxiliary problems $P_q$, namely, if $F(q^*) = 0$ then $q^*$ is the optimal value of $FP$ and the optimal control and the optimal trajectory of $P_q$ are optimal also for problem $FP$ (see [9], Theorem 4.6.1 p. 157). It follows that to find the maximum of problem $FP$ is equivalent to determine the root of equation

$$F(q) = 0 \quad .$$

Hence, following Dinkelbach's idea [3], the solution of the original fractional problem $FP$ could be obtained by means of an iterative procedure which starts from a given value of $q$ such that $F(q) \geq 0$; at each iteration the value of $q$ increases, determining a sequence of values of $F(q)$ that converges to zero.

The effectiveness of this method depends of course on the features of the auxiliary optimal control problems. Given the special nature of the linear problem $FP$, it is possible to find the explicit expression of function $F(q)$. As it will become clear in Section 4, this sharply reduces the difficulty of the problem and allows to find its solution $q^*$ solving a single equation.

### 3.1 Optimal advertising policies for problem $FP$

The first three propositions characterize the optimal advertising policies for problem $FP$. In particular, Proposition 2 details case (a) of Proposition 1 and Proposition 3 restates part of propositions 1 and 2 in terms of the parameters of the model $FP$ and not in terms of the optimal value of the problem.

**Proposition 1** *Let be $q^*$ the optimal value of problem $FP$. Then the following statements hold:*

*(a) if $q^* \neq k\epsilon/\delta$ then there exists a unique optimal control of problem $FP$ and this optimal control is bang-bang with at most one switch;*

*(b) if $q^* = k\epsilon/\delta$ and $\delta > k$ then the optimal control of $FP$ is $a^*(t) = \overline{a}\ \forall\ t \in [0, T]$;*

*(c) if $q^* = k\epsilon/\delta$ and $\delta < k$ then the optimal control of $FP$ is $a^*(t) = 0\ \forall\ t \in [0, T]$;*

*(d) if $q^* = k\epsilon/\delta$ and $\delta = k$ then any control function $a(t) \in Adv$ is optimal for $FP$.*

*Proof.* See Appendix. ◇

**Proposition 2** *Let be $q^* \neq k\epsilon/\delta$. Then the following statements hold:*

*(i) if $\delta > k$ then it is optimal to advertise at the end of the selling period;*

*(ii) if $\delta < k$ then it is optimal to advertise at the beginning of the selling period;*

*(iii) if $\delta = k$ and $q^* < \epsilon$ then the optimal control of $FP$ is $a^*(t) = \overline{a}\ \forall\ t \in [0,T]$;*

*(iv) if $\delta = k$ and $q^* > \epsilon$ then the optimal control of $FP$ is $a^*(t) = 0\ \forall\ t \in [0,T]$.*

*Proof.* See Appendix ◇

**Proposition 3** *If $\delta \neq k$ or $A_0 + bkT \neq \epsilon C_0$ then there exists a unique optimal control of problem $FP$ and this optimal control is bang-bang. Otherwise (i.e. $\delta = k$ and $A_0 + bkT = \epsilon C_0$) any control function $a(t) \in Adv$ is optimal.*

*Proof.* See Appendix. ◇

### 3.2 Optimal control of problem $P_q$

It is possible to analyze the optimal solutions of problem $P_q$ for any fixed value of $q$. In particular we show that if the optimal control of problem $P_q$ has exactly one switch, then this switching time is

$$\tau = T + \frac{1}{\delta}\ \ln \frac{k\epsilon - \delta q}{\epsilon(k-\delta)}\ . \tag{11}$$

and we can obtain the explicit form of the auxiliary function $F(q)$. To simplify notation in the next propositions, let us define:

$$L = \left(1 - \frac{k}{\delta}\right) e^{-\delta T} + \frac{k}{\delta}\ , \tag{12}$$

and observe that

$$if \quad \delta > k \quad then \quad L \in \left(\frac{k}{\delta}\ ,\ 1\right), \tag{13}$$

$$if \quad \delta < k \quad then \quad L \in \left(1,\ \frac{k}{\delta}\right). \tag{14}$$

Of course, if $\delta = k$ then $L = 1$.

**Proposition 4** *The following statements hold.*
*(a) Let $\delta > k$; the optimal control $a(t)$ of problem $P_q$ has the following form:*

$$\begin{array}{rll} if \quad q \leq \epsilon L & then \quad a(t) = \overline{a}\ \forall\ t \in (0,T); & \\ if \quad q \in (\epsilon L, \epsilon) & then \quad a(t) = \begin{cases} 0, & if\ \ t \in (0,\tau); \\ \overline{a}, & if\ \ t \in (\tau, T); \end{cases} & \\ if \quad q \geq \epsilon & then \quad a(t) = 0\ \forall\ t \in (0,T). & \end{array}$$

*(b) Let $\delta < k$. The optimal control $a(t)$ of problem $P_q$ has the following form:*

$$\begin{array}{rll} if \quad q \leq \epsilon & then & a(t) = \overline{a} \ \forall \ t \in (0,T); \\ if \quad q \in (\epsilon, \epsilon L) & then & a(t) = \left\{ \begin{array}{ll} \overline{a}, & if \ \ t \in (0,\tau); \\ 0, & if \ \ t \in (\tau, T); \end{array} \right. \\ if \quad q \geq \epsilon L & then & a(t) = 0 \ \forall \ t \in (0,T). \end{array}$$

*(c) Let $\delta = k$. The optimal control $a(t)$ of problem $P_q$ has the following form:*

$$\begin{array}{rll} if \quad q < \epsilon & then & a(t) = \overline{a} \ \forall \ t \in (0,T); \\ if \quad q > \epsilon & then & a(t) = 0 \ \forall \ t \in (0,T); \\ if \quad q = \epsilon & then & a(t) \ \ is \ \ any \ \ from \ \ Adv. \end{array}$$

*Proof.* See Appendix. ◇

We can observe that the switching time (11) is "well-defined". Indeed, if $\delta > k$ and $q \in (\epsilon L, \epsilon)$ then $(k\epsilon - \delta q)(k - \delta) > 0$ due to (13) while if $\delta < k$ and $q \in (\epsilon, \epsilon L)$ then, due to (14), again $(k\epsilon - \delta q)(k - \delta) > 0$.

From the above proposition we can distinguish two main cases. If the decay rate of goodwill is high, thus meaning that the advertising forgetfulness is high enough (this situation corresponds to case $\delta > k$), the optimal advertising policy $a(t)$ has in general the following structure

$$a(t) = \left\{ \begin{array}{ll} 0, & \text{if} \ \ t \in (0,\tau) \\ \overline{a}, & \text{if} \ \ t \in (\tau, T) \end{array} \right.$$

namely, it is convenient to make no advertising initially, whereas it is convenient to undertake maximum advertising at the end of the communication period.
On the other hand, if $\delta < k$, that is the decay rate of goodwill is low, it is convenient to maximize the advertising effort from the very first and the optimal advertising policy $a(t)$ has the following form

$$a(t) = \left\{ \begin{array}{ll} \overline{a}, & \text{if} \ \ t \in (0,\tau) \\ 0, & \text{if} \ \ t \in (\tau, T) \end{array} \right. .$$

### 3.3 Description of function $F(q)$

We derive now the explicit expression of function $F(q)$ with the aim to obtain an explicit solution of problem $FP$.

**Proposition 5** *The following statements hold:*

*(a) if $\delta > k$ then*

$$F(q) = \begin{cases} F_1(q), & \text{if } \; q \leq \epsilon L \; ; \\ F_2(q), & \text{if } \; \epsilon L < q < \epsilon \; ; \\ F_3(q), & \text{if } \; q \geq \epsilon \; ; \end{cases} \tag{15}$$

*(b) if $\delta < k$ then*

$$F(q) = \begin{cases} F_1(q), & \text{if } \; q \leq \epsilon \; ; \\ F_4(q), & \text{if } \; \epsilon < q < \epsilon L \; ; \\ F_3(q), & \text{if } \; q \geq \epsilon L \; ; \end{cases} \tag{16}$$

*(c) if $\delta = k$ then*

$$F(q) = \begin{cases} F_1(q), & \text{if } \; q \leq \epsilon \; ; \\ F_3(q), & \text{if } \; q \geq \epsilon \; ; \end{cases} \tag{17}$$

*where*

$$F_1(q) = -(C_0 + \overline{a}T)q + A_0 L + bkT + \frac{\epsilon \overline{a}}{\delta}(1 - L + kT), \tag{18}$$

$$F_2(q) = -\left(C_0 + \frac{\overline{a}}{\delta}\right) q + A_0 L + bkT + \frac{\epsilon \overline{a}}{\delta}\left[1 + \frac{\delta q - k\epsilon}{\delta \epsilon} \ln \frac{\delta q - k\epsilon}{\epsilon(\delta - k)}\right], \tag{19}$$

$$F_3(q) = -C_0 q + A_0 L + bkT, \tag{20}$$

$$F_4(q) = -\left(C_0 - \frac{\overline{a}}{\delta}\right) q + A_0 L + bkT - \frac{\epsilon \overline{a}}{\delta}\left\{L + \frac{\delta q - k\epsilon}{\delta \epsilon}\left[\delta T + \ln \frac{\delta q - k\epsilon}{\epsilon(\delta - k)}\right]\right\}. \tag{21}$$

*Proof.* See Appendix. ◇

It is interesting to note that functions $F_1(q)$ and $F_3(q)$ are linear: this property will be used in the algorithm proposed in section 4.

## 4 An algorithm to solve problem $FP$

Dinkelbach's approach for fractional optimal control problems requires to solve equation

$$F(q) = 0$$

usually by means of a numerical approach. Function $F$ is usually known only implicitly and each step of the solution procedure requires to solve an optimal control problem (see [9]).

Nevertheless, fortunately, according to Proposition 5, the structure of problem $FP$ allows to give the explicit expression of function $F$ for each $q$. This expression is obtained by solving the linear optimal control problem $P_q$ depending on the parameter $q$. This sharply reduces the difficulty of the problem and allows to find its solution $q^*$ solving a single equation.

As a consequence of Proposition 5 and using the monotonicity properties of Dinkelbach's function $F(q)$ it is possible to propose the following algorithm in order to find the solution of equation $F(q) = 0$ thus solving problem $FP$ .

***Statement of the algorithm***

*The optimal value $q^*$ and the optimal control $a^*$ of Problem $FP$ can be found as follows:*

*if* $\delta > k$ *then*
*if* $F_1(\epsilon L) \leq 0$ *then* $F_1(q^*) = 0$ *and* $a^*(t) = \overline{a}\ \forall\ t \in (0, T)$
*else if* $F_3(\epsilon) \geq 0$ *then* $F_3(q^*) = 0$ *and* $a^*(t) = 0\ \forall\ t \in (0, T)$
*else* $F_2(q^*) = 0$ *and* $a^*(t) = \begin{cases} 0 & \textit{if}\ t \in (0, \tau^*) \\ \overline{a} & \textit{if}\ t \in (\tau^*, T) \end{cases}$

*if* $\delta < k$ *then*
*if* $F_1(\epsilon L) \leq 0$ *then* $F_1(q^*) = 0$ *and* $a^*(t) = \overline{a}\ \forall\ t \in (0, T)$
*else if* $F_3(\epsilon) \geq 0$ *then* $F_3(q^*) = 0$ *and* $a^*(t) = 0\ \forall\ t \in (0, T)$
*else* $F_4(q^*) = 0$ *and* $a^*(t) = \begin{cases} \overline{a} & \textit{if}\ t \in (0, \tau^*) \\ 0 & \textit{if}\ t \in (\tau^*, T) \end{cases}$

*if* $\delta = k$ *then*
*if* $F_1(\epsilon) = F_3(\epsilon) = 0$ *then any control function* $a(t) \in Adv$ *is optimal*
*else if* $F_1(\epsilon) \leq 0$ *then* $F_1(q^*) = 0$ *and* $a^*(t) = \overline{a}\ \forall\ t \in (0, T)$
*else* $F_3(q^*) = 0$ *and* $a^*(t) = 0\ \forall\ t \in (0, T)$

*where* $\tau^*$ *is defined by (11).*

Remark that to solve equations $F_2(q^*) = 0$ and $F_4(q^*) = 0$ it is possible to apply some well known numerical solution techniques, e.g. a Newton-like method, due to the smoothness of functions $F_2$ and $F_4$: both these functions are decreasing, convex and $C^\infty$ in the intervals $(\epsilon L, \epsilon)$ and $(\epsilon, \epsilon L)$, respectively. Remark that $q^* > 0$ since it is the optimal value of the efficiency ratio of problem $FP$.

## 4.1 Sensitivity analysis

It is possible to study the sensitivity of the optimal value of problem $FP$ with respect to changes in the parameters of the problem. By means of the implicit function theorem we can obtain the derivative of the optimal value $q^*$, with respect to each parameter.

In fact, the optimal value $q^*$ is implicitly defined by the following equation

$$F(q^*) = 0 \ .$$

When $\delta \neq k$ function $F$ is differentiable and decreasing, therefore $\partial F/\partial q < 0$ and it is possible to apply the implicit function theorem to obtain the sign of the derivative of $q^*$ with respect to the parameters. It is thus possible to prove that:

$$\frac{\partial q^*}{\partial b} > 0 \ ; \quad \frac{\partial q^*}{\partial \delta} < 0 \ ; \quad \frac{\partial q^*}{\partial A_0} > 0 \ ; \quad \frac{\partial q^*}{\partial C_0} < 0 \ ; \quad \frac{\partial q^*}{\partial \overline{a}} \geq 0 \ ; \quad \frac{\partial q^*}{\partial k} > 0 \ ; \quad \frac{\partial q^*}{\partial \epsilon} \geq 0 \ .$$

### 4.2 A numerical example

Consider the case $\overline{a} = 30$, $k = 3$, $b = 0.10$, $T = 1$, $\delta = 4$, $\epsilon = 2$, $A_0 = 0.1$, $C_0 = 1$. Since $\delta > k$ we have case $(b)$ of Propositions 4 and 5, the optimal control will be $0 - \overline{a}$. Function $F$ turns out to be

$$F(q) = \begin{cases} -31q + 49.05677 & \text{if } \ q \leq 1.50916 \\ -8.5q + 15.37546 + 1.875(4q-6)log(2q-3) & \text{if } \ 1.50916 < q < 2 \\ -q + 0.37546 & \text{if } \ q \geq 2 \end{cases} \ . \qquad (22)$$

In Figure 1 we plot function $F(q)$. The optimum value is defined by $F_2(q^*) = 0$, i.e.

$$-8.5q^* + 15.37546 + 1.875(4q^* - 6)log(2q^* - 3) = 0 \ .$$

A simple numerical computation allows to find $q^* \simeq 1.64960$. From (11) we obtain the optimal switch time $\tau^* \simeq 0.69834$.

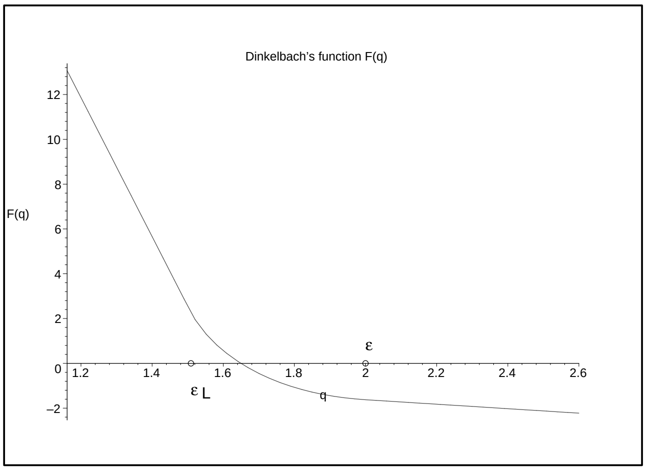


Figure 1: Graph of Dinkelbach's function.

## 5 Conclusions

In this paper we consider an advertising efficiency maximization problem. The problem turns out to be a linear fractional optimal control problem; to solve it we propose to use the Dinkelbach approach and the particular structure of the functional allows to obtain an almost explicit solution of the problem and, in particular, to determine how the structure of the optimal advertising policies changes depending on goodwill's decay.

The same methodology could be applied to a more general class of fractional functionals: this will be addressed in the next future research.

Moreover, the same efficiency index considered in the objective functional of the advertising problem could be used to compare different advertisers or different media in a Data Envelopment Analysis (DEA) framework. This could therefore lead to a dynamic approach to DEA, which will be another stimulating topic for future research.

## 6 Appendix

In this appendix we give the proofs of propositions 1-5.

### 6.1 Proof of Proposition 1

We prove that the Proposition holds for any value of $q$ and therefore also for the optimal value $q^*$.

Given problem $P_q$ consider its Hamiltonian function

$$H_q = kA(t) - qa(t) + p(t)[-\delta A(t) + \epsilon a(t)] = [k - p(t)\delta]A(t) + [p(t)\epsilon - q]a(t),$$

where, due to Pontryagin Maximum Principle, function $p(t)$ is such that

$$\begin{cases} \dot{p}(t) = \delta p(t) - k; \\ p(T) = 1; \end{cases}$$

i.e.

$$p(t) = \left(1 - \frac{k}{\delta}\right) e^{\delta(t-T)} + \frac{k}{\delta} \ .$$

Therefore, the switching function is

$$G_q(t) = p(t)\epsilon - q = \frac{\epsilon(\delta - k)}{\delta}\ e^{\delta(t-T)} + \frac{\epsilon k}{\delta} - q \ . \tag{23}$$

(a) Let $q \neq k\epsilon/\delta$. If $\delta = k$ then function (23) is constant and non-zero: positive if $\epsilon > q$ and negative if $\epsilon < q$; while if $\delta \neq k$ then function (23) is not constant and has at most one zero. So if $q \neq k\epsilon/\delta$ then function (23) has at most one zero. Therefore there exists a unique optimal control of problem $P_q$ and this optimal control is bang-bang with at most one switch.

(b) Let $q = k\epsilon/\delta$ and $\delta > k$. Then function (23) is

$$G_q(t) = \frac{\epsilon(\delta - k)}{\delta}\ e^{\delta(t-T)} \tag{24}$$

and is positive $\forall\, t \in [0,T]$. Therefore the optimal control of $P_q$ is $a(t) = \overline{a}\ \forall t \in [0,T]$.

(c) Let $q = k\epsilon/\delta$ and $\delta < k$. Then function (23) has form (24) and is negative $\forall\, t \in [0,T]$. Therefore the optimal control of $P_q$ is $a(t) = 0\ \forall t \in [0,T]$.

(d) Let $q = k\epsilon/\delta$ and $\delta = k$. Then function (23) is identically zero. Therefore in this case it is not possible to apply the Pontryagin Maximum Principle. But, fortunately, the problem $P_q$ can be solved by another way. To do this, let us first obtain the following auxiliary lemma.

**Lemma 1** *Let $q = k\epsilon/\delta$ and $\delta = k$. Then $F(q) = A_0 + kbT - \epsilon C_0$.*

*Proof of Lemma 1.* Using the motion equation of problem $P_q$ we can rewrite its objective function this way (remark that now $q = \epsilon$)

$$F(q) = \left(A(T) + k\int_0^T (A(t)+b)dt\right) - q\left(C_0 + \int_0^T a(t)dt\right) =$$

$$= \int_0^T [kA(t) - qa(t) + kb]dt + A(T) - qC_0 =$$

$$= \int_0^T [kA(t) - qa(t)]dt + \int_0^T \dot{A}(t)dt + A_0 + kbT - qC_0 =$$

$$= \int_0^T [kA(t) - qa(t) - \delta A(t) + \epsilon a(t)]dt + A_0 + kbT - qC_0 =$$

$$= A_0 + kbT - \epsilon C_0\ .$$

◇

Now we can complete the proof of case (d) of the Proposition. Due to Lemma 1, function $F(q)$ is constant. Therefore in this case any control function $a(t) \in Adv$ is optimal for $P_q$.

◇

## 6.2 Proof of Proposition 2

Also in this case, we prove that the Proposition holds for any value of $q$ and therefore also for the optimal value $q^*$.

Due to case (a) of Proposition 1, there exists a unique optimal control of problem $P_q$ and this optimal control is bang-bang with at most one switch. It means that the optimal control $a(t)$ can be only of form $(0 - \overline{a})$ or of form $(\overline{a} - 0)$.

(a) Let $\delta > k$. Then the switching function (23) increases. Therefore, the optimal control $a(t)$ has form $(0 - \overline{a})$.

(b) Let $\delta < k$. Then the switching function (23) decreases. Therefore, the optimal control $a(t)$ has form $(\overline{a} - 0)$.

(c) Let $\delta = k$ and $q < \epsilon$. Then the switching function (23) is constant and positive. Therefore, the optimal control of problem $FP$ is $a(t) = \overline{a}\ \forall\, t \in [0,T]$.

(d) Let $\delta = k$ and $q > \epsilon$. Then the switching function (23) is constant and negative. Therefore, the optimal control of problem $FP$ is $a(t) = 0\ \forall\, t \in [0,T]$.

◇

### 6.3 Proof of Proposition 3

Let be $q^*$ such that $F(q^*) = 0$.

If $q^* = k\epsilon/\delta$ and $\delta = k$ then any control function $a(t) \in Adv$ is optimal, see case (d) of Proposition 1; moreover $F(q^*) = A_0 + bkT - \epsilon C_0$, see Lemma 1 in the proof of Proposition 1. Since $F(q^*) = 0$ then $A_0 + bkT = \epsilon C_0$. Therefore, if $A_0 + bkT \neq \epsilon C_0$ then either $q^* \neq k\epsilon/\delta$ or $\delta \neq k$; in the first case, i.e. $q^* \neq k\epsilon/\delta$, the optimal control of Problem $FP$ is bang-bang due to the case (a) of Proposition 1.

If instead $q^* = k\epsilon/\delta$ and $\delta \neq k$ then the optimal control is either $a^*(t) = \overline{a}\ \forall\ t \in [0, T]$ or $a^*(t) = 0\ \forall\ t \in [0, T]$ (see cases (b) and (c) of Proposition 1), i.e. the control is again bang-bang.

$\diamond$

### 6.4 Proof of Proposition 4

Consider again function (23), i.e. the switching function of problem $P_q$ :

$$G_q(t) = \frac{\epsilon(\delta - k)}{\delta}\ e^{\delta(t-T)} + \frac{\epsilon k}{\delta} - q\ .$$

One has

$$G_q(t) = 0\ \Leftrightarrow\ t = T + \frac{1}{\delta}\ \ln \frac{k\epsilon - \delta q}{\epsilon(k - \delta)}\ .$$

In particular, it means that $G_q(t)$ can be equal to zero only if $(k\epsilon - \delta q)(k - \delta) > 0$. Moreover, we can understand when this (unique!) zero $\tau$ (see (11)) lies in interval $(0, T)$. Indeed,

$$0 < \tau < T \quad \Leftrightarrow \quad e^{-\delta t} < \frac{k\epsilon - \delta q}{\epsilon(k - \delta)} < 1. \tag{25}$$

If $\delta > k$ and $k\epsilon < \delta q$ then (25) gives us (recall that $L$ is defined in (12))

$$0 < \tau < T \quad \Leftrightarrow \quad \epsilon L < q < \epsilon,$$

while if $\delta < k$ and $k\epsilon < \delta q$ then (25) gives

$$0 < \tau < T \quad \Leftrightarrow \quad \epsilon < q < \epsilon L.$$

Finally, if $\delta > k$ then $k\epsilon < \delta\epsilon L$ due to (13), while if $\delta < k$ then $k\epsilon > \delta\epsilon L$ due to (14). Summarizing the above considerations, we obtain the following properties.

1) Let $\delta > k$. Then function $G_q(t)$ is strictly increasing. Moreover

$$\begin{array}{rcl} \textit{if} \quad q < \epsilon L & \textit{then} & G_q(t) > 0\ \forall\ t \in (0, T); \\ \textit{if} \quad q \in (\epsilon L, \epsilon) & \textit{then} & G_q(t) \left\{ \begin{array}{ll} < 0, & \text{if}\ \ t \in (0, \tau); \\ > 0, & \text{if}\ \ t \in (\tau, T); \end{array} \right. \\ \textit{if} \quad q > \epsilon & \textit{then} & G_q(t) < 0\ \forall\ t \in (0, T). \end{array}$$

Therefore case (a) of the Proposition is proved.

2) Let $\delta < k$. Then function $G_q(t)$ is strictly decreasing. Moreover

$$\begin{array}{rcl} if \;\; q<\epsilon & then & G_q(t)>0 \; \forall \; t\in(0,T); \\ if \;\; q\in(\epsilon,\epsilon L) & then & G_q(t)\left\{ \begin{array}{ll} >0, & \text{if } \; t\in(0,\tau); \\ <0, & \text{if } \; t\in(\tau,T); \end{array}\right. \\ if \;\; q>\epsilon L & then & G_q(t)<0 \; \forall \; t\in(0,T). \end{array}$$

Therefore, case (b) of the Proposition is proved.

3) Let $\delta = k$. Then $G_q(t) \equiv \epsilon - q$, i.e. function $G_q(t)$ is constant: positive if $\epsilon > q$ and negative if $\epsilon < q$. Moreover, if $q = \epsilon$ then, due to Lemma 1 (see proof of Proposition 1), function $F(q)$ is constant, so any control function $a(t) \in Adv$ is optimal for Problem $P_q$. Therefore, case (c) of the Proposition is proved. $\diamond$

## 6.5 Proof of Proposition 5

(a) Let $\delta > k$. Then optimal control $a(t)$ of Problem $P_q$ is as in case (a) of Proposition 4. Let us substitute a(t) in the motion equation and solve it. This way we receive the state variable (goodwill) $A(t)$.

If $q \leq \epsilon L$ then

$$A(t) = \left(A_0 - \frac{\epsilon\overline{a}}{\delta}\right) e^{-\delta t} + \frac{\epsilon\overline{a}}{\delta} \; , \tag{26}$$

if $q \in (\epsilon L, \epsilon)$ then

$$A(t) = \left\{ \begin{array}{ll} A_0 e^{-\delta t}, & \text{if } \; t\in(0,\tau); \\ \\ \left(A_0 - \frac{\epsilon\overline{a}}{\delta} e^{\delta\tau}\right) e^{-\delta t} + \frac{\epsilon\overline{a}}{\delta}, & \text{if } \; t\in(\tau,T); \end{array}\right. \tag{27}$$

if $q \geq \epsilon$ then

$$A(t) = A_0 e^{-\delta t} \; . \tag{28}$$

Substituting (26), (27) and (28) in $F(q)$ we receive, respectively, (18), (19) and (20).

(b) Let $\delta < k$. Then optimal control $a(t)$ of Problem $P_q$ is as in case (b) of Proposition 4. Similarly to case (a), we can receive that if $q \leq \epsilon$ then $A(t)$ is (26); if $q \in (\epsilon, \epsilon L)$ then

$$A(t) = \left\{ \begin{array}{ll} \left(A_0 - \frac{\epsilon\overline{a}}{\delta}\right) e^{-\delta t} + \frac{\epsilon\overline{a}}{\delta} \; , & \text{if } \; t\in(0,\tau); \\ \\ \left[A_0 - \frac{\epsilon\overline{a}}{\delta}\left(1 - e^{-\delta\tau}\right)\right] e^{-\delta t} \; , & \text{if } \; t\in(\tau,T); \end{array}\right. \tag{29}$$

while if $q \geq \epsilon L$ then $A(t)$ is (28). Substituting (26), (29) and (28) in $F(q)$ we receive, respectively, (18), (21) and (20).

(c) Let $\delta = k$. Then the optimal control $a(t)$ of Problem $P_q$ is as in case (c) of Proposition 4. Therefore, if $q < \epsilon$ then $A(t)$ is (26); while if $q > \epsilon$ then $A(t)$ is (28). Substituting (26), and (28) in $F(q)$ we receive, respectively, (18) and (20). To end the proof, it is sufficient to remark that in this case (i.e. when $\delta = k$) if $q = \epsilon$ then $F_1(q) = F_3(q)$. $\diamond$

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